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RHEOLOGY OF CONCENTRATED MIXTURES OF FLUID WITH SMALL PARTICLES. PARAMETERS OF PHASE INTERACTION

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The combined motion of fluid and solid, fluid, or gaseous particles suspended in it is considered. Particles are assumed to be small and to be distributed in space at random, hence the Reynolds numbers of flows past particles of various dimensions are small, and we can neglect any random ("quasi-turbulent") pulsations of the two phases. Parameters defining phase interaction under conditions of unsteady nonuniform flow are calculated.

Disperse systems consisting of a fluid and particles suspended in it may be arbitrarily separated into two categories. In "soft" systems which belong to the first category the particles and the fluid are in regular and defined motion; random pulsations of phases are weak and do not appreciably affect the system behavior, so that in the majority of cases these may be neglected. Systems of the second category ("rigid" systems) are, on the other hand, subject to intensive chaotic pulsation of particles and fluid, which have a decisive effect on the rheological properties and distinctive features of the transport process in such systems.

A consistent hydromechanical theory of "soft" systems has been developed only for fairly small concentrations of the disperse system, so that we can either neglect particle interaction altogether, or make an approximate allowance for pair-interaction between these [1]. The theory of random quasi-turbulent motion of phases and of its effect on the rheology of "rigid" systems is presented in [3]. However the neglect in the latter of phase pulsations made it necessary to specify a priori certain important parameters which determine the behavior of a disperse system. Hence the development of the theory for highly concentrated soft systems is of considerable interest, also in the analysis of processes in rigid disperse systems.

The problem of constrained motion of fluid in concentrated cloud of random distributed particles and of resulting phase interaction was considered earlier on various assumptions [4 - 6].

A stricter treatment of the problem of steady flow through a lattice of solid particles was given by Tam [7] who used the approximation of "point" forces whereby perturbations generated by point forces applied to the fluid at the centers of particles are substituted for perturbations introduced in the stream by particles. This method, which is free of arbitrary empirical assumptions, was extended to unsteady flows in [8].

Below we formulate a method of successive approximations in which the point-force approximation proposed in [7, 8] is taken as a reasonable "zero" approximation, thus providing in principle the possibility of improving the accuracy of results derived with the use of this approximation. Results obtained in [7, 8] are extended to streams of fluid which not only are unstable but also nonuniform and the particles are not necessarily solid or stationary.

1. Basic assumptions and equations. Let us consider the flow of fluid in a polydisperse cloud consisting of fine spherical particles such that the Reynolds numbers defining the flow past particles of various dimensions are small, and we can use the Stokes form of the linearized equations of hydromechanics.

Owing to the linearity of these equations, the local velocity $\mathbf{V}(t, \mathbf{r})$ and pressure $P(t, \mathbf{r})$ of the fluid in the interstices between particles can be presented in the form

$$\begin{gathered}\mathbf{V}(t, \mathbf{r}) = \mathbf{V}_0(t, \mathbf{r}) + \sum_{j=1}^{N} \mathbf{V}(t, \mathbf{r}; \mathbf{r}^{(j)}) \\ P(t, \mathbf{r}) = P_0(t, \mathbf{r}) + \sum_{j=1}^{N} P(t, \mathbf{r}; \mathbf{r}^{(j)})\end{gathered} \tag{1.1}$$

where $\mathbf{V}_0(t, \mathbf{r})$ and $P_0(t, \mathbf{r})$ are the velocity and pressure of the unperturbed flow, while $\mathbf{V}(t, \mathbf{r}; \mathbf{r}^{(j)})$ and $P(t, \mathbf{r}; \mathbf{r}^{(j)})$ are the perturbations created by a particle with its center at point $\mathbf{r}^{(j)}$, and N is the total number of particles in the system. In addition to (1.1) we also consider the velocity and pressure fields of the fluid perturbed by all particles except the j-th (1.2)

$$\mathbf{V}^{(j)'}(t, \mathbf{r}; \mathbf{r}^{(j)}) = \mathbf{V}(t, \mathbf{r}) - \mathbf{V}(t, \mathbf{r}; \mathbf{r}^{(j)}), \quad P^{(j)'}(t, \mathbf{r}; \mathbf{r}^{(j)}) = P(t, \mathbf{r}) - P(t, \mathbf{r}; \mathbf{r}^{(j)})$$

The quantities defined by (1.2), unlike those defined by (1.1), have no singularities at point $\mathbf{r} = \mathbf{r}^{(j)}$ (see the reasoning in [7]).

The fields defined by (1.1) and (1.2) satisfy equations

$$\begin{gathered}d_0(\partial / \partial t)\, \mathbf{V}_0(t, \mathbf{r}) = -\nabla P_0(t, \mathbf{r}) + \mu_0 \Delta \mathbf{V}_0(t, \mathbf{r}) - \nabla\Phi(t, \mathbf{r}) \\ d_0(\partial / \partial t)\, \mathbf{V}(t, \mathbf{r}; \mathbf{r}^{(j)}) = -\nabla P(t, \mathbf{r}; \mathbf{r}^{(j')}) + \mu_0 \Delta \mathbf{V}(t, \mathbf{r}, \mathbf{r}^{(j)}) - \mathbf{F}(t, \mathbf{r}; \mathbf{r}^{(j)}) \\ \mathbf{F}(t, \mathbf{r}; \mathbf{r}^{(j)}) = \int \mathbf{p}(t, \mathbf{r})\, \delta(\mathbf{r} - \mathbf{r}^{(j)} - \mathbf{a}^{(j)})\, d\mathbf{a}^{(j)} \\ \nabla \mathbf{V}_0(t, \mathbf{r}) = 0, \quad \nabla \mathbf{V}(t, \mathbf{r}; \mathbf{r}^{(j)}) = 0\end{gathered} \tag{1.3}$$

where d_0 and μ_0 are the density and viscosity of the fluid, $\mathbf{p}(t, \mathbf{r})$ is the density vector of particle surface stress and $\mathbf{a}^{(j)}$ is a vector drawn from the center of the j-th particle to any arbitrary point of its surface. Quantities $\mathbf{r}^{(j)}$ depend on time. Vector $\mathbf{F}(t, \mathbf{r}; \mathbf{r}^{(j)})$ denotes the total reaction of the fluid flowing around the j-th particle, and $\Phi(t, \mathbf{r})$ is the potential of external mass forces. Integration in (1.3) is carried out over the surface of the j-th particle. The equations for the summations in (1.1) and (1.2) follow from (1.3).

An explicit determination of fluid motion in terms of quantities (1.1) is not only unnecessary but also essentially impossible, since the position of particles in the system is to a great extent random and unknown (regularly packed particulate layers are the only exception). Hence it is reasonable to follow [7, 8] and consider only certain mean characteristics of fluid flow in particle interstices, introducing for this purpose the concept of particle ensemble constituting the cloud and, also, certain specific assumptions about the properties of this ensemble.

Let us assume that the particles are statistically independent and any cross-correlation between their position in space is absent. In this case the ensemble distribution function of N particles in terms of their radii $a^{(j)}$ and of radius vectors of their centers $\mathbf{r}^{(j)}$ ($j = 1, 2, \ldots, N$) can be represented as a simple superposition of unitary distribution functions equal for all particles. We have

$$\varphi_N(t, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(N)};\ a^{(1)}, \ldots, a^{(N)}) = \prod_{j=1}^{N} \varphi(t, \mathbf{r}^{(j)}, a^{(j)}) \qquad (1.4)$$

We assume for definiteness that the distribution functions appearing in (1.4) are normalized with respect to unity in regions of their respective definition. They depend on t as on a parameter.

Strictly speaking, the superposition relationship (1.4) is valid only in the case of statistically independent point-particles. Centers of particles of finite volume cannot lie arbitrarily close to each other, which is not allowed for in (1.4). The related error increases with increasing volume concentration ρ of particles in the system, and for ρ close to the concentration ρ_* of a tightly packed system. Relationship (1.4) must be replaced by a more precise one which includes binary distribution functions, usual in statistical physics of liquids and dense gases. Binary distribution functions must, also, be introduced whenever there exist stable correlation links between particles of the system (e.g., when longlived doublets of particles are generated). Relationship (1.4) yields however, entirely satisfactory results for statistically independent particles up to $\rho \approx 0.5$. Hence we limit our considerations to these particle concentrations and shall use for simplicity the relationship (1.4) without further reservations.

In addition to these functions we can, also, introduce denumerable (numerical) and volume concentrations of particles by means of equality

$$n(t, \mathbf{r}) = N \int \varphi(t, \mathbf{r}, a)\, da, \qquad \rho(t, \mathbf{r}) = \frac{4}{3}\pi N \int a^3 \varphi(t, \mathbf{r}, a)\, da \qquad (1.5)$$

We define the operation of averaging over the ensemble of particles in the following manner:

$$\langle f \rangle = \int \cdots \int f \varphi_N(t;\ \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(N)};\ a^{(1)}, \ldots, a^{(N)}) d\mathbf{r}^{(1)} \ldots d\mathbf{r}^{(N)} da^{(1)} \ldots da^{(N)} \qquad (1.6)$$

where f is an arbitrary function. Functions which are independent of $\mathbf{r}^{(j)}$ and $a^{(j)}$ are, obviously, not affected by such averaging. From (1.1) and (1.2) we obtain, in particular

$$\langle \mathbf{V} \rangle = \mathbf{V}_0(t, \mathbf{r}) + \langle \mathbf{V}(t, \mathbf{r};\ \mathbf{r}^{(j)}) \rangle N, \quad \langle P \rangle = P_0(t, \mathbf{r}) + \langle P(t, \mathbf{r};\ \mathbf{r}^{(j)}) \rangle N \quad (1.7)$$
$$\langle \mathbf{V}^{(j)\prime} \rangle = \mathbf{V}_0 + \langle \mathbf{V}(t, \mathbf{r};\ \mathbf{r}^{(j)}) \rangle (N - 1), \quad \langle P^{(j)\prime} \rangle = P_0 + \langle P(t, \mathbf{r};\ \mathbf{r}^{(j)}) \rangle (N - 1)$$

From (1.7) we obtain the equalities

$$\langle \mathbf{V} \rangle = \langle \mathbf{V}^{(j)\prime} \rangle, \qquad \langle P \rangle = \langle P^{(j)\prime} \rangle \qquad (1.8)$$

which are asymptotically valid for $N \gg 1$.

Using operator (1.6), from (1.3) we also obtain the averaged equations

$$d_0 \langle (\partial / \partial t) \mathbf{V} \rangle = - \langle \nabla P \rangle + \mu_0 \langle \Delta \mathbf{V} \rangle - \nabla \Phi - N \langle \mathbf{F} \rangle = 0, \quad \langle \nabla \mathbf{V} \rangle = 0 \quad (1.9)$$

The particles (if solid) are generally subject to translational and rotational motions about axes passing through their centers. We denote the velocity of translational motion of the center of the j-th particle and its angular velocity by $\mathbf{W}^{(j)}(t)$ and $\mathbf{\Omega}^{(j)}(t)$, respectively. According to (1.4) and (1.6), these two quantities are not affected by the averaging over the ensemble. The translational and rotational velocities of particles can, also, be defined by vector functions $\mathbf{W}$ $(t, \mathbf{r}, a)$ and Ω $(t, \mathbf{r}, a)$ such that

$$\mathbf{W}(t, \mathbf{r}^{(j)}, a^{(j)}) \equiv \mathbf{W}^{(j)}(t), \qquad \Omega(t, \mathbf{r}^{(j)}, a^{(j)}) \equiv \Omega^{(j)}(t)$$

However, unlike $\mathbf{W}^{(j)}(t)$ and $\Omega^{(j)}(t)$, these vector functions are affected by the averaging over the ensemble. If there are no quasiturbulent pulsations, which were considered in [3], or they are weak (which we assume), these functions can be considered to be regular.

For the purpose of this analysis it is sufficient to consider $\mathbf{W}$ $(t, \mathbf{r}, a)$ and Ω $(t, \mathbf{r}, a)$ and, also φ $(t, \mathbf{r}, a)$ as certain known quantities specified a priori. This is entirely adequate for the flow inside a cloud of particles considered in [7, 8]. In the general case these quantities represent solutions of certain equations (not considered here) which describe the mean motions in a soft disperse system in the approximation of interpenetrating and interacting continuous media.

The order of magnitude of the space scale l of significant variation of $\mathbf{V}$ $(t, \mathbf{r})$ and P $(t, \mathbf{r})$ coincides with the mean distance between particle centers. The space scale of related magnitudes averaged over the ensemble, as well as that of φ $(t, \mathbf{r}, a)$; $\mathbf{W}$ $(t, \mathbf{r}, a)$ and Ω $(t, \mathbf{r}, a)$ and, generally, of all parameters defining the motions of the disperse system phases is, however, not necessarily equal l Assuming for simplicity that the scale of all averaged parameters are of equal orders of magnitude and denoting these by L, we set

$$L \gg l \sim a\rho^{-1/3} \gtrsim a, \qquad \varepsilon_l = l / L \ll 1$$

This assumption implies that we can chose a small physical volume of mixture which would contain a number of particles sufficient for averaging over the ensemble, and such that all averaged parameters in this volume can be considered as virtually independent of coordinates. Note that the existence of such volume is a necessary condition of admissibility of using the methods of mechanics of continuous media for describing the average motions in a disperse system, when this is assumed to be a superposition of interpenetrating and interacting continua.

The time-scale of variation of $\mathbf{V}$ $(t, \mathbf{r})$ and P $(t, \mathbf{r})$ and of related averaged magnitudes are the same and are, for example, defined by the dynamics of boundary condition variation imposed on the system. Let the order of magnitude of this scale be equal τ. The time-scale T of significant variation of the distribution function φ $(t, \mathbf{r}, a)$ must considerably exceed τ, since that variation is related not to the variation of the local hydrodynamic pattern in the neighborhood of an arbitrary particle, but to the redistribution of particles throughout the whole volume of the disperse system, i.e., in a volume whose order of magnitude is L. Hence we assume

$$T \gg \tau, \qquad \varepsilon_\tau = \tau / T \ll 1$$

For simplicity we assume henceforth that ε_l and ε_τ are small magnitudes of the same order, i.e., $\varepsilon_l \sim \varepsilon_\tau \sim \varepsilon$. It is convenient to use parameter ε in the derivation of asymptotic solutions of above equations.

In a small neighborhood $| \mathbf{r} - \mathbf{r}_0 | \lesssim l$ of an arbitrary point r_0 all parameters which define the average motion can be expressed by series expansions in ε. We shall need in the following the expansion

$$W(t, r, a) = W_0(t, r_0, a) + \sum_{m=1}^{\infty} \varepsilon^m W_m(t, r - r_0, a;\ r_0), W_m = O(W_0) \quad (1.10)$$

where the summation contains coefficients of Taylor's series. In region $t - t_0 \lessgtr \tau$, $|r - r_0| \lessgtr l$ we similarly have

$$\varphi(t, r, a) = \varphi_0(t_0, r_0, a) + \sum_{m=1}^{\infty} \varepsilon^m \varphi_m(t - t_0,\ r - r_0,\ a;\ t_0, r_0)$$
$$\int \varphi_0 da = \frac{n(t_0, r_0)}{N}, \qquad \int \varphi_m da = O\left(\frac{n(t_0, r_0)}{N}\right) \quad (1.11)$$

Expansion

$$\varphi_N(t; r^{(1)}, \ldots, r^{(N)};\ a^{(1)}, \ldots, a^{(N)}) = \sum_{m=0}^{\infty} \varepsilon^m \varphi_{Nm}, \ \varphi_{N0} = \prod_{j=1}^{N} \varphi_0(t_0, r_0, a^{(j)})$$

$$\varphi_{N1} = \sum_{m=1}^{N} \varphi_1(t - t_0, r^{(m)} - r_0, a^{(m)};\ t_1, r_0) \prod_{j=1,\ j\neq m}^{N} \varphi_0(t_0, r_0, a^{(j)}) \quad (1.12)$$

corresponds to formula (1.11).

If (1.12) is used for averaging a certain function f in accordance with formula (1.6), then the m-th term appearing in the expansion of $\langle f\rangle$ is of the order of $(\varepsilon N)^m$, i.e., the subsequent terms of this expansion for $N \to \infty$ are not necessarily small in comparison with the preceding terms. If, however, the averaged function depends on the coordinates of the particle centers and on the radii of $\sim n(t_0, r_0)\, l^3$ particles contained in $\sim l^3$ volume surrounding point r_0, then, as can be readily shown, the m-th term is of the order of $(\varepsilon n(t_0, r_0)\, l^3)^m \sim \varepsilon^m$, i.e., the series for $\langle f\rangle$ is an asymptotic expansion.

2. Expansion in multipoles and the condition of self-consistency. It is shown in [7, 8] and, also, by the analysis given below that the flow of fluid in a cloud of particles is subjected to effective hydrodynamic screening of each particle by its closest neighbors, in the sense that the velocity of that particle and the velocity and pressure of the fluid around it are substantially affected only by those neighboring particles whose distance from it is of the order of $l \sim a\rho^{-1/3}$. Hence, by using for averaging over the ensemble the distribution function (1.12) and taking into consideration Sect. 1, we obtain

$$\left\langle \frac{\partial V}{\partial t} \right\rangle = \frac{\partial \langle V\rangle_0}{\partial t} - \varepsilon \left(\int \cdots \int \frac{\partial \varphi_{N_1}}{\partial t} V(t, r)\, dr^{(1)} \ldots dr^{(N)} da^{(1)} \ldots da^{(N)} + \right.$$
$$\left. + O(\varepsilon)\right) = \left(\frac{\partial}{\partial t}\right) \langle V\rangle_0 + O(\varepsilon) \quad (2.1)$$

and, similarly,

$$\langle \nabla P\rangle = \nabla \langle P\rangle_0 + O(\varepsilon), \qquad \langle \Delta V\rangle = \Delta \langle V\rangle_0 + O(\varepsilon)$$
$$\langle \nabla V\rangle = \nabla \langle V\rangle_0 + O(\varepsilon) \quad (2.2)$$

The subscript zero at angle brackets denotes here and in the following an averaging by function φ_{N0} defined in (1.12). Explicit expressions for magnitudes denoted in (2.2) by $O(\varepsilon)$ are not given owing to their bulkiness.

Let us examine $\langle F\rangle$ appearing in (1.9). Using the definition of vector $F(t, r;\ r^{(j)})$ in (1.3), we expand the delta-function in the integral into a series in powers of components of vector $a^{(j)}$. We obtain

$$\mathbf{F}(t, \mathbf{r}; \mathbf{r}^{(j)}) = \int \mathbf{p}(\mathbf{r}^{(j)} + \mathbf{a}^{(j)})\,\delta(\mathbf{r} - \mathbf{r}^{(j)} - \mathbf{a}^{(j)})\,d\mathbf{a}^{(j)} =$$

$$= \int \mathbf{p}(\mathbf{r}^{(j)} + \mathbf{a}^{(j)}) \left(\sum_{q=1}^{\infty} \frac{1}{q!} (\mathbf{a}^{(j)}\nabla)^q \delta(\mathbf{r} - \mathbf{r}^{(j)}) \right) d\mathbf{a}^{(j)}$$

Altering the order of summation and integration, we obtain

$$F_i(t, \mathbf{r}; \mathbf{r}^{(j)}) = \sum_{q=1}^{\infty} {}^q G_{ik\ldots m}(\mathbf{r}^{(j)}) \int \frac{\partial}{\partial r_k} \cdots \frac{\partial}{\partial r_m} \delta(\mathbf{r} - \mathbf{r}^{(j)}) \tag{2.3}$$

$$^qG_{ik\ldots m}(\mathbf{r}^{(j)}) = \frac{1}{q!} \int \mathbf{p}(\mathbf{r}^{(j)} + \mathbf{a}^{(j)})\, a_k^{(j)} \ldots a_m^{(j)} d\mathbf{a}^{(j)}$$

where ${}^q\mathrm{G}(\mathbf{r}^{(j)})$ are tensors of rank q. This expansion is essentially an expansion in multipoles and analogous in its meaning to expansions in the potential theory. The first term (the monopole) in (2.3) defines the force applied at point $\mathbf{r} = \mathbf{r}^{(j)}$ equal in magnitude to the exerted on this particle by the fluid flowing around it. The second term is determined by tensor ${}^2\mathrm{G}(\mathbf{r}^{(j)})$ which is the point "dipole moment" of stresses distributed over the particle surface. The point-force approximation introduced in [7, 8],obviously, takes into account only the first term in (2.3).

The definition of tensors ${}^q\mathrm{G}(\mathbf{r}^{(j)})$ evidently implies that $\langle {}^{q+1}G \rangle \sim a \langle {}^qG \rangle$ for any q, ${}^qG_{i\ldots j}$. It is also clear that in accordance with Sect. 1 we have $(\partial / \partial r_k) \ \langle {}^qG \rangle \sim L^{-1}$ $\langle {}^qG \rangle$. This makes it possible to prove that the order of each subsequent terms with respect to ε in the expansion of $\langle \mathrm{F} \rangle$ is by a unity higher than that of the preceding one. Let us prove this for the first two terms. From (2.3) we have

$$\langle {}^1\mathrm{G}(\mathbf{r}^{(j)})\,\delta(\mathbf{r} - \mathbf{r}^{(j)}) \rangle = \int {}^1\mathrm{G}(\mathbf{r}^{(j)})\,\delta(\mathbf{r} - \mathbf{r}^{(j)})\,\varphi(t, \mathbf{r}^{(j)}, a^{(j)})\,d\mathbf{r}^{(j)} da^{(j)} + O(\varepsilon) =$$

$$= \left(\frac{n}{N}\right) \langle {}^1\mathrm{G} \rangle + O(\varepsilon)$$

and, integrating by parts, yields

$$\langle {}^2\mathrm{G}(\mathbf{r}^{(j)})\,\nabla\delta(\mathbf{r} - \mathbf{r}^{(j)}) \rangle = \int {}^2\mathrm{G}(\mathbf{r}^{(j)})\,\varphi(t, \mathbf{r}^{(j)}, a^{(j)})\,\nabla\delta(\mathbf{r} - \mathbf{r}^{(j)})\,dr^{(j)} da^{(j)} +$$

$$+ O(\varepsilon) = -\left(\frac{n}{N}\right) \nabla \langle {}^2\mathrm{G} \rangle + O(\varepsilon) \sim \varepsilon \left(\frac{n}{N}\right) \langle {}^1\mathrm{G} \rangle$$

The extension to terms of higher order is elementary.

Thus, in accordance with (2.3) vector $\langle \mathbf{F} \rangle$ is expressed by a series in powers of ε, whose coefficients, determined by averaging over the ensemble (1.2), also depend on ε, and which is a generalized series expansion, as defined by Erdélyi. Using this series and representing $\langle \mathbf{V} \rangle$ and $\langle P \rangle$ in the form of series

$$\langle \mathbf{V} \rangle = \sum_{m=0}^{\infty} \varepsilon^m \langle \mathbf{V} \rangle^{(m)}, \qquad \langle P \rangle = \sum_{m=0}^{\infty} \varepsilon^m \langle P \rangle^{(m)} \tag{2.4}$$

with unknown coefficients, and considering (2.1) and (2.2), in which $\langle \mathbf{V} \rangle$ and $\langle P \rangle$ are defined by (2.4) as generalized asymptotic expansions, we can obtain from (1.9) a system of successive approximation equations for determining all coefficients of series (2.4). We limit our analysis to the zero approximation only in which

$$d_0 (\partial / \partial t) \langle \mathbf{V} \rangle^{(0)} = -\nabla \langle P \rangle^{(0)} + \mu_0 \Delta \langle \mathbf{V} \rangle^{(0)} - \nabla\Phi - n \langle {}^1\mathrm{G} \rangle_0^{(0)} = 0 \tag{2.5}$$

$$\nabla \langle \mathbf{V} \rangle^{(0)} = 0$$

where $\langle {}^1\mathrm{G} \rangle_0^{(0)}$ is the principal term of the asymptotic expansion of $\langle {}^1\mathrm{G} \rangle$. The equations of subsequent approximations can be derived in a similar manner. In Eqs. (2.5) of the zero approximation the point-forces are of the form given in [7, 8], and are based on the assumptions of locally-uniform stationary ensemble of particles, as defined by the distribution function (1.12).

It is expedient to use Eqs. (2.5) in a system of coordinates attached to the center of a certain j-th particle. We have

$$d_0(\partial/\partial t)\,\mathbf{U}^{(j)} = -\nabla\langle P\rangle^{(0)} + \mu_0\Delta\mathbf{U}^{(j)} - \nabla\Phi - d_0(\partial/\partial t)\,\mathbf{W}^{(j)} - n\langle{}^1\mathbf{G}\rangle_0^{(0)} = 0$$

$$\nabla\mathbf{U}^{(j)} = 0, \qquad \mathbf{U}^{(j)} = \langle\mathbf{V}\rangle^{(0)} - \mathbf{W}^{(j)}, \qquad \Delta\mathbf{U}^{(j)} \equiv \Delta\langle\mathbf{V}\rangle^{(0)} \tag{2.6}$$

To simplify the subsequent analysis we apply to (2.6) the Fourier transformation and, as the result, obtain the following equations:

$$i\omega d_0\mathbf{U}_\omega^{(j)} = -\nabla P_\omega + \mu_0\Delta\mathbf{U}_\omega - \nabla\Phi_\omega - id_0\omega\mathbf{W}_\omega^{(j)} - n\mathbf{G}_\omega = 0$$

$$\nabla\mathbf{U}_\omega^{(j)} = 0 \tag{2.7}$$

where ω is the frequency, the subscript ω denotes Fourier transformations of respective quantities and angle brackets and other subscripts at pressure and force symbols have been omitted for simplicity.

Equations (2.6) and (2.7) may, in particular, be used for investigating the flow around the j-th particle. The effect of all remaining particles on the formation of velocity and pressure fields in the neighborhood of that particle, i.e., the influence of the constrained flow, is taken into account by the introduction of volume force $n\langle{}^1\mathbf{G}\rangle_0^{(0)}$ (or of its Fourier component $n\mathbf{G}_\omega$) which defines the drag of particles in the stream of fluid.

The quantity $\mathbf{G}_\omega$ must, obviously, be a linear combination of linearly-independent vectors which define the unperturbed by the j-th particle flow at point $\mathbf{r} = \mathbf{r}^{(j)}$ (this follows directly from the linearity of the equations of motion). There are only two such vectors $\mathbf{U}_\omega^{(j)\prime} = \langle\mathbf{V}^{(j)\prime}\rangle_\omega^{(0)} - \mathbf{W}_\omega^{(j)} = \langle\mathbf{V}\rangle_\omega^{(0)} - \mathbf{W}_\omega^{(j)} = \mathbf{U}_\omega^{(j)}$, $\quad \Delta\mathbf{U}_\omega^{(j)\prime} = \Delta\mathbf{U}_\omega^{(j)}$

(relationship (1.8) has been used here; vector $\nabla\langle P^{(j)}\rangle_\omega^{(0)} = \nabla\langle P\rangle_\omega^{(0}$ is linearly dependent on $\mathbf{U}_\omega^{(j)}$ and $\Delta\mathbf{U}_\omega^{(j)}$ which defines their relationship is determined by the equations of motion). Hence we can write

$$\mathbf{G}_\omega^{(j)} = D'(\omega, a^{(j)})\,\mathbf{U}_\omega^{(j)} + D''(\omega, a^{(j)\prime})\,\Delta\mathbf{U}_\omega^{(j)} \tag{2.8}$$

which is a natural generalization of formulas proposed in [7, 8] for the determination of force. Obviously, $D'' \sim a^2 D'$, i.e., the ratio of the second term to the first is of the order of ε^2. Hence in Eqs. (2.7) we have to take into consideration only the first term of (2.8). We then have (*)

$$n\mathbf{G}_\omega = n\langle D'\mathbf{U}_\omega^{(j)}\rangle_0 = \mu_0\alpha\mathbf{U}_\omega^{(j)} + \mu_0\mathbf{\Gamma}^{(j)} \tag{2.9}$$

$$\mu_0\alpha = n\int D'\varphi_0\,da, \qquad \mu_0\mathbf{\Gamma}^{(j)} = \mu_0\alpha\mathbf{W}_\omega^{(j)} - n\int D'\mathbf{W}_0\varphi_0\,da$$

where $\mathbf{W}_0$ is that defined in (1.10). Finally, substituting (2.9) into (2.7), we obtain

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*) The expressions for $\mathbf{G}_\omega^{(j)}$, used in [7, 8] are somewhat different. It can be shown, however, that this difference is of a higher order with respect to ε and, consequently, it is immaterial in the zero approximation.

$$(\Delta - \beta^2)\mathbf{U} = \mu_0^{-1}\nabla(P + \Psi), \qquad \nabla\mathbf{U} = 0 \tag{2.10}$$
$$\beta^2 = \mu_0^{-1}(\mu_0\alpha + id_0\omega), \qquad \Psi = \Phi + id_0\omega\mathbf{W}\mathbf{r} + \mathbf{\Gamma}\mathbf{r}$$

The indices ω and (i) have been omitted here for simplicity and Ψ is the effective potential of mass forces acting on the fluid. The first term in the expression for Ψ defines the external mass field and the second that of inertia forces in the chosen system of coordinates. The third term defines the additional field of mass forces whose presence is due to the difference of velocities of particles of various radii and, consequently also to that of forces and their interaction with the fluid (particles moving relative to the separated one entrain also the fluid). This term is specific to polydisperse systems, for monodisperse systems $\Gamma \equiv 0$.

Equation (2.10) may be formally considered to be the equation defining the motion of certain imaginary fluid subjected throughout the space to the action of the volume "friction" force $\mu_0\alpha\mathbf{U}$. This motion simulates the true flow of fluid in particle interstices. In a sense it has a certain, although very superficial, similarity with the concept of an imaginary homogeneous medium, suggested on purely experimental considerations in [4].

Equations (2.10) contain the unknown parameters α and Γ which can be determined from the condition of the self-consistency theory of the form considered in [7, 8]. Namely, by solving the problem of flow around a particle defined by Eqs. (2.10) it is easy to calculate the force (2.8) and thus find the expression for the coefficient D' which depends on α as a parameter. Using (2.9) for calculating α, we derive an algebraic (transcendental) equation for the complex parameter α whose solution makes it possible to close completely the theory. In fact, parameter Γ which remains undetermined can be calculated, if α ,determined by formula (2.9), is known.

3. Calculation of the force and moment acting on a particle.

Let us consider the flow around a certain sample particle placed in a fluid containing other particles, on the assumption that the flow is defined by Eqs. (2.10). Let the velocity and pressure of the stream unperturbed by that particle be $\mathbf{U}_0(\mathbf{r})e^{i\omega t}$ and $P_0(\mathbf{r})e^{i\omega t}$. Setting the coordinate origin at the center of the solid spherical particle, we obtain for the amplitude, velocity and pressure the following problem:

$$(\Delta - \beta^2)\mathbf{U} = \mu_0^{-1}\nabla R, \qquad \nabla\mathbf{U} = 0, \qquad R = P + \Psi \tag{3.1}$$
$$\mathbf{U} = \mathbf{\Omega} \times \mathbf{r} \quad (r = a); \qquad \mathbf{U} \to \mathbf{U}_0 \quad (r \to \infty)$$

For the flow around a liquid (gaseous) sphere we similarly have the problem

$$(\Delta - \beta^2)\mathbf{U}^{(0)} = \mu_0^{-1}\nabla R^{(0)}, \qquad \nabla\mathbf{U}^{(0)} = 0, \qquad R^{(0)} = P^{(0)} + \Psi$$
$$(\Delta - \gamma^2)\mathbf{U}^{(1)} = \mu_0^{-1}\nabla R^{(1)}, \qquad \nabla\mathbf{U}^{(1)} = 0, \qquad R^{(1)} = P^{(1)} + \Phi + id_0Wr \tag{3.2}$$
$$\gamma^2 = id_1\mu_1^{-1}\omega; \qquad \mathbf{U}^{(0)} = \mathbf{U}^{(1)}, \quad (\sigma^{(0)}\mathbf{n})_\tau = (\sigma^{(1)}\mathbf{n})_\tau \quad (r = a)$$
$$\mathbf{U}^{(0)} \to \mathbf{U}_0 \quad (r \to \infty); \qquad \mathbf{U}^{(1)} = O(1), \qquad R^{(1)} = O(1) \quad (r \to 0)$$

The indices zero and unity relate to regions outside and inside the particle, respectively; σ is the stress tensor; $\mathbf{n}$ is a unit vector of the normal to the particle surface, and subscript τ denotes a tangential component of stress.

The condition of continuity of normal stresses at the particle surface can be, also, readily derived. However, this condition which is necessary for determining small deviations of the drop or bubble shape from spherical, is entirely immaterial in the solution of the hydrodynamic problem of flow around such particles.

Let us represent $\mathbf{U}_0(\mathbf{r})$ and $R_0(\mathbf{r})$ in the form of expansions in terms of basic vector functions constructed on spherical functions

$$\mathbf{U}_0(\mathbf{r}) = \sum_{m=0}^{\infty} F_m \frac{\mathbf{r}}{r} s_m + G_m r \nabla s_m + H_m \mathbf{r} \times \nabla s_m, \quad R_0(\mathbf{r}) = \mu_0 \sum_{m=0}^{\infty} L_m s_m \quad (3.3)$$

We seek the solutions of problems (3.1) or (3.2) in the form of expansions similar to (3.3) with coefficients $f_m^{(0)}$, $g_m^{(0)}$, $h_m^{(0)}$ and $l_m^{(0)}$ outside the particle and $f_m^{(1)}$, $g_m^{(1)}$, $h_m^{(1)}$ and $l_m^{(1)}$ inside it. In these expansions s_m is a spherical function of the m-th order containing $2m+1$ terms corresponding to the principal and the associated Legendre functions. For example, the symbol $A_m s_m$ denotes the summation

$$A_m s_m = A_m^{(0)} P_m + \sum_{m'=1}^{m} A_{m+}^{(m')} P_m^{(m')} \cos m'\varphi + A_{m-}^{(m')} P_m^{(m')} \sin m'\varphi$$

and so on. The coefficients in (3.3) and in the expansions for solving (3.1) and (3.3) depend only on r.

Substituting expansions (3.3) into (3.1), using Euler's theorem on homogeneous functions, and taking into consideration that $\Delta s_m = -m(m+1)r^{-2}s_m$, we obtain for the coefficients in (3.3) the system of equations

$$\begin{gathered}
F_m'' + \frac{2}{r} F_m' - \frac{m(m+1)+2}{r^2} F_m - \beta^2 F_m + \frac{2m(m+1)}{r^2} G_m - L_m' = 0 \\
G_m'' + \frac{2}{r} G_m' - \frac{m(m+1)}{r^2} G_m - \beta^2 G_m + \frac{2}{r^2} F_m - \frac{1}{r} L_m = 0 \qquad (3.4) \\
F_m' + \frac{2}{r} F_m - \frac{m(m+1)}{r^2} G_m = 0, \\
H_m'' + \frac{2}{r} H_m' - \frac{m(m+1)}{r^2} H_m - \beta^2 H_m = 0
\end{gathered}$$

(a prime denotes here differentiaition with respect to r). The coefficients $f_m^{(0)}$, $g_m^{(0)}$, $h_m^{(0)}$ and $l_m^{(0)}$ satisfy similar equations, while the coefficients $f_m^{(1)}$, $g_m^{(1)}$, $h_m^{(1)}$ and $l_m^{(1)}$ satisfy equations which differ from (3.4) by the substitution of γ^2 for β^2.

The boundary conditions of problem (3.1) imposed for $r = a$ yield the following relationships between the coefficients:

$$f_m^{(0)} + F_m = 0 \qquad g_m^{(0)} + G_m = 0, \qquad h_m^{(0)} + H_m = -a\Omega\delta_{1m} \qquad (3.5)$$

From the boundary conditions of problem (3.2) we, similarly, obtain

$$\begin{gathered}
f_m^{(0)} + F_m = f_m^{(1)} = 0 \qquad g_m^{(0)} + G_m = g_m^{(1)}, \qquad h_m^{(0)} + H_m = h_m^{(1)} \\
g_m^{(0)\prime} + G_m' = \varkappa g_m^{(1)\prime} + a^{-1}(1-\varkappa) g_m^{(1)} \qquad (3.6) \\
h_m^{(0)\prime} + H_m' = \varkappa h_m^{(1)\prime} + a^{-1}(1-\varkappa) h_m^{(1)}, \qquad \varkappa = \mu_1/\mu_0
\end{gathered}$$

Equations (3.1) and (3.2) readily yield $\Delta R = 0$ and $\Delta R^{(k)} = 0$ $(k = 0, 1)$ from which follows that $l_m^{(0)} = (m_m^{(0)} r^m + n_m^{(0)} r^{-m-1})\beta^2$

$$L_m = (M_m + N_m r^{-m-1})\beta^2, \qquad l_m^{(1)} = (m_m^{(1)} r^m + n_m^{(1)} r^{-m-1})\gamma^2$$

Substituting this and the expression for G_m defined by the third of Eqs. (3.4) into the first of these, we obtain

$$F_m'' + \frac{4}{r} F_m' - \frac{(m-1)(m+2)}{r^2} F_m - \beta^2 F_m = m\beta^2 M_m r^{m-1} - \frac{m+1}{r^{m+2}} \beta^2 N_m$$

The solution of this equation is the sum of partial and general solutions of the related homogeneous equation. After calculation, we have

$$F_m = A_m S_m(r) + B_m Q_m(r) - mM_m r^{m-1} + (m+1) N_m r^{-m-2}$$

$$S_m = 2^m \zeta^{m-1} \frac{d^m}{d(\zeta^2)^m} \frac{\operatorname{sh}\zeta}{\zeta}, \qquad Q_m = (-2)^m \zeta^{m-1} \frac{d^m}{d(\zeta^2)^m} \frac{e^{-\zeta}}{\zeta} \tag{3.7}$$

$$\zeta = \beta r$$

For $m = 0$ there is no finite solution at point $r = 0$. For $m \geqslant 1$ functions $S_m(r)$ in (3.7) are free of singularities at that point. Functions $Q_m(r)$ which are divergent for $r \to 0$ vanish for $r \to \infty$. From the third of Eqs. (3.4) we obtain

$$G_m = \frac{2}{m(m+1)} A_m \left(S_m + \frac{r}{2} S_m'\right) + \frac{2}{m(m+1)} B_m \left(Q_m + \frac{r}{2} Q_m'\right) - - M_m r^{m-1} - N_m r^{-m-2} \qquad (m \neq 0)$$

The solution of the last of Eqs. (3.4) is

$$H_m = C_m r S_m + D_m r Q_m$$

Taking into consideration the conditions of boundedness of solution defined by (3.1) and (3.2), we obtain the most general admissible expressions for the unknown coefficients of the form $F_m = A_m S_m - mM_m r^{m-1}, \quad f_m^{(1)} = a_m^{(0)} S_m - m m_m^{(1)} r^{m-1}$

$$G_m = \frac{2}{m(m+1)} A_m \left(S_m + \frac{r}{2} S_m'\right) - M_m r^{m-1}, \qquad H_m = C_m r S_m$$

$$g_m^{(1)} = \frac{2}{m(m+1)} a_m^{(1)} \left(S_m + \frac{r}{2} S_m'\right) - m_m^{(1)} r^{m-1}, \qquad h_m^{(1)} = c_m^{(1)} r S_m \tag{3.8}$$

$$f_m^{(0)} = b_m^{(0)} Q_m - (m+1) m_m^{(0)} r^{-m-2}, \qquad h_m^{(0)} = d_m^{(0)} r Q_m$$

$$g_m^{(0)} = \frac{2}{m(m+1)} b_m^{(0)} \left(Q_m + \frac{r}{2} Q_m'\right) - n_m^{(0)} r^{-m-2}, \qquad l_m^{(0)} = \beta^2 n_m^{(0)} r^{-m-1}$$

$$L_m = \beta^2 M_m r^m, \qquad l_m^{(1)} = \gamma^2 m_m^{(1)} r^m \qquad (m \geqslant 1)$$

where A_m, C_m, M_m, $b_m^{(0)}$, $d_m^{(0)}$, $n_m^{(0)}$, $a_m^{(1)}$, $c_m^{(1)}$ and $m_m^{(1)}$ are constants and A_m, C_m and M_m define the flow unperturbed by the sample particle and are specified a priori while the remaining constant are determined by Eqs. (3.5) or (3.6). In the definition of the first six coefficients S_m and Q_m are functions of βr, and in that of the last three they are functions of γr.

In the absence of an external force field of potential Ψ the stresses in an area element with the normal vector $\mathbf{n} = \mathbf{r} / r$ in field (3.3) are defined by

$$\sigma \mathbf{n} = \mu_0 \sum_{m=0}^{\infty} \left[(2F_m' - L_m) \frac{\mathbf{r}}{r} s_m + \left(G_m' - \frac{G_m}{r} + \frac{F_m}{r}\right) r \nabla s_m + \right.$$

$$\left. + \left(H_m' - \frac{H_m}{r} \mathbf{r} \times \nabla s_m\right) \right] \tag{3.9}$$

The first term in (3.9) defines normal stresses and the two last ones the tangential stresses. The formulas of stresses in fields differing from (3.3), by the substitution of F_m, G_m, H_m and L_m for $f_m^{(0)}$, $g_m^{(0)}$, $h_m^{(0)}$ and $l_m^{(0)}$ are of a similar form. To calculate the forces and moments acting on the particle when $\Psi = 0$ it is necessary to integrate

$\sigma \mathbf{n}$ in (3.9) or $\mathbf{r} \times \sigma \mathbf{n}$ over the particle surface, a sphere of radius a.

Note that the terms in expressions of the kind of (3.9) are proportional to vectors $\mathbf{r} \times \nabla s_m$, vanish in the integration, while among the remaining terms in (3.9) only those which correspond to $m = 1$ are nonzero. Thus, when considering the reaction of the stream on the body, it is sufficient to determine in (3.8) only the constants denoted by the subscript $m = 1$. Constants A_1, C_1 and M_1 which define the unperturbed flow, can be readily expressed in terms of $\mathbf{U}_0(\mathbf{r})$ and of its derivative at point $r = 0$.

After calculations and certain transformations of the expressions for force $\mathbf{F}$ and moment $\mathbf{M}$ acting on a rigid particle for $\Psi = 0$, we obtain

$$\mathbf{F} = 6\pi\mu_0 a(1 + \xi + {}^1/_3\xi^2)\mathbf{U}_0(0) - 2\pi\mu_0 a^3[1 - 3\xi^{-2}(e^{\xi} - 1 - \xi)]\Delta\mathbf{U}_0(0) \qquad (3.10)$$

$$M = 4\pi\mu_0 a^3 e^{\xi}(1 + \xi)^{-1} \operatorname{rot} \mathbf{U}_0(0) - 8\pi\mu_0 a^3(1 + \xi + {}^1/_3\xi^2)(1 + \xi)^{-1}\Omega$$

$$\xi = \beta a$$

We recall that here we consider the amplitudes of forces and moments acting on a particle in a harmonic stream. In an arbitrary unstable flow these are botained from the corresponding amplitudes defined below and by the inverse Fourier transformation.

The total force $\mathbf{F}_t$ exerted on a solid particle by the stream flowing around it and by the pressure field Ψ (see (2.9)) is of the form

$$\mathbf{F}_t = \mathbf{F} - {}^4/_3\pi a^3(\nabla\Phi + id_0\omega\mathbf{W} + \Gamma) \qquad (3.11)$$

The second term of (3.11) represents the total Archimedean force which acts on a particle in the "external" field with potential Ψ and is the resultant of forces due to the actual external field Φ and of inertia in the chosen system of coordinates with the addition of Γ to the mass forces, specific to polydisperse systems.

Formulas (3.10) and (3.11) contain, as particular cases, all known formulas for $\mathbf{F}$ and $\mathbf{M}$ acting on a single particle in a steady or harmonic stream at low Reynolds numbers. For $\beta = 0$ we obtain the known Faxen formula for the force in a stationary inhomogeneous stream, and for $\beta = id_0\omega\mu_0^{-1}$ and $\mathbf{U}_0 = -\mathbf{W} = \text{const}$ the known expression for the force acting on a single particle harmonically oscillating in a stationary fluid [9], etc. Note that besides the force defined by (3.11) the particle is subjected to the forces of the external mass field and of inertia.

After calculations, for the force and the moment acting on a liquid sphere we obtain the expressions

$$\mathbf{F} = 6\pi\mu_0 a[(1 + \xi)q + {}^1/_3\xi^2]\mathbf{U}_0(0) - 2\pi\mu_0 a^3[1 - 3\xi^{-2}(e^{\xi} - 1 - \xi)q]\Delta\mathbf{U}_0(0)$$

$$\mathbf{M} = 4\pi\mu_0 a^3 \frac{\varkappa\eta S_2(\eta)}{\xi^3[\xi S_1(\eta)Q_2(\xi) + \varkappa\eta Q_1(\xi)S_2(\eta)]} \operatorname{rot} \mathbf{U}_0(0) \qquad (3.12)$$

$$q = \frac{\varkappa\eta^2 S_1(\eta) + 2(1 - \varkappa)\eta S_2(\eta)}{\varkappa\eta^2 S_1(\eta) + [1 + \xi + 2(1 - \varkappa)]\eta S_2(\eta)} \qquad (\eta = \gamma a)$$

Functions S_m and Q_m were defined in (3.7). The formula for $\mathbf{F}$ in (3.12) differs from the corresponding formula in (3.10) only by the presence of the factor q. The expressions for $\mathbf{F}$ and $\mathbf{M}$ acting on an isolated drop are readily derived from (3.12). For example it is not difficult to obtain for $\xi = \eta = 0$ a generalization of the Faxen formula which was analyzed in [10]. Relationship (3.11) remains valid for $\mathbf{F}_t$.

Formulas (3.10) – (3.12) completely define the forces exerted by the stream on a solid or liquid particle, only if a or the parameter $\xi = \beta a$ which depends on a and appears in these formulas. Below we consider the determination of the latter parameter for suspensions of solid particles and emulsions of drops or bubbles.

Let us, for simplicity, consider a monodisperse system of particles (analysis of a polydisperse system with an arbitrary function of particle distribution with respect to their dimensions does not present any fundamental complications). In this case

$$\mu_0 \alpha = nD'(a) \tag{3.13}$$

where n is the denumerable concentration of particles. To determine ξ we use the condition of self-consistency which in accordance with (3.10), (3.12), (3.13) and Sect. 2 assumes the form

$$6\pi\mu_0 a\,[(1+\xi)q + {}^1\!/_3\xi^2] = D'(a) \tag{3.14}$$

where for solid particles $q = 1$ and for drops and bubbles q depends on ξ as defined in (3.12). Multiplying both parts of (3.14) by the volume concentration ρ and using (3.13) we obtain

$$6\pi\mu_0 a\rho\,[(1+\xi)q + {}^1\!/_3\xi^2] = {}^4\!/_3\pi\mu_0\,(\alpha a^2)a$$

Expressing α in terms of β and then in terms of ξ, we obtain

$$\alpha a^2 = \xi^2 - i\omega', \quad \omega' = \omega\,/\,\omega_0, \quad \omega_0 = \nu_0\, a^{-2}, \quad \nu_0 = \mu_0\,/\,d_0$$

With the use of these relationships, we obtain for ξ the equation

$$(1 - {}^3\!/_2\rho)\xi^2 = i\omega' + {}^9\!/_2\rho\,(1+\xi)q \tag{3.15}$$

For solid particles (3.15) is a quadratic equation in ξ whose solution is of the form

$$\xi = {}^3\!/_2\,(2-3\rho)^{-1}\,\{3\rho + [8\rho - 3\rho^2 + {}^8\!/_9 i\,(2-3\rho)\omega']^{1/2}\} \tag{3.16}$$

and completely defined the quantities (3.10) and (3.11). For $\omega' = 0$ we have the known result [7].

For a liquid or gaseous particle Eq. (3.15) derived from (3.12) is a cubic equation and, as can readily be shown, has a unique root with q as its positive real part. It can, also, be shown that for $\omega' = 0$ this root is always smaller than the value of ξ for solid particles, as defined by formula (3.16). In the particular case of $\omega' \to 0$ from (3.15) we obtain

$$\xi^2 = \frac{9\rho}{2-3\rho}\,\frac{(2+3\varkappa)(1+\xi)}{\xi+3+3\varkappa} \tag{3.17}$$

When, on the other hand, $\omega' \to \infty$ the liquid and gaseous particles begin to behave (as regards their interaction with the stream) as solid particles, and this statement is the more accurate the greater are $\varkappa$ and $\lambda = d_1\,/\,d_0$.

For $\varkappa \to \infty$ (i.e., for a steady flow of solid particle suspensions) from (3.12) we obtain for the moment the following expression:

$$\mathbf{M} = 4\pi\mu_0 a^3 \frac{i\omega'\lambda e^{\xi}\,\mathrm{rot}\,\mathbf{U}_0(0)}{15\,(1+\xi+{}^1\!/_2\xi^2) + i\omega\lambda\,(1+\xi)}\,, \qquad \lambda = \frac{d_1}{d_0}$$

In a steady flow $\mathbf{M} = 0$, and a particle rotates with the angular velocity (see (3.10))

$$\Omega = \frac{1}{2}\,\frac{e^{\xi}}{1+\xi+{}^1\!/_3\xi^2}\,\mathrm{rot}\,\mathbf{U}_0(0) \tag{3.18}$$

We stress that this quantity depends not only on the curl of velocity of the unperturbed fluid flow at the point occupied by the particle center but, also, on ξ which, in turn depends to a great extent on particle concentration in the system and on physical parameters of phases. This very important dependence is altogether omitted in the formu-

-lation of equations of the phase moment of momentum conservation for a disperse system considered as a medium with inner rotation.

We note in conclusion that the proposed method of analysis of flows of soft disperse systems makes it possible to effect further averaging over the system volume with the view of obtaining a closed system of equations for "macroscopic" parameters which define the average motion of phases, as interpenetrating and interacting continuous media (see the derivation and discussion of modified Darcy equations in [8]).

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